

# Exts and Vertex Operators

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## Abstract

The direct product of two Hilbert schemes of the same surface has natural K-theory classes given by the alternating Ext groups between the two ideal sheaves in question, twisted by a line bundle. We express the Chern classes of these virtual bundles in terms of Nakajima operators.

## 1 Introduction

### 1.1

Let  $S$  be a nonsingular quasi-projective algebraic surface. The Hilbert scheme  $\mathrm{Hilb}(S, n)$  of  $n$  points on  $S$  has been the focus of numerous recent studies, see e.g. [2, 4, 10] for a survey. In particular, the cohomology of  $\mathrm{Hilb}(S, n)$  has been described using certain operators acting on

$$\mathcal{F} = \bigoplus_n \mathcal{F}_n, \quad \mathcal{F}_n = H^*(\mathrm{Hilb}(S, n), \mathbb{Q}), \quad (1)$$

introduced by Nakajima [9] and Grojnowski [3].

In this paper we consider another natural set of operators,  $W(\mathcal{L})$ , depending on a line bundle  $\mathcal{L}$  on  $S$ . These operators act on  $\mathcal{F}$ , and are defined in terms of Chern classes of sheaves of Ext-groups. We prove an explicit formula for  $W(\mathcal{L})$  in terms of the Nakajima operators. It describes  $W(\mathcal{L})$  as a *vertex operator* acting on  $\mathcal{F}$ .

Similar operators may be defined and studied for more general moduli spaces of sheaves. In this paper we will be content with the case of rank-one

torsion-free sheaves on a surface (i.e. the Hilbert schemes of points), leaving generalizations to future papers.

The main application we have in mind concerns the case of  $T$ -equivariant cohomology of  $\text{Hilb}(\mathbb{C}^2, n)$ , with respect to the natural action of  $T \cong (\mathbb{C}^\times)^2$  on  $\mathbb{C}^2$ . In this case, the trace of  $\mathbf{W}(\mathcal{L})$  becomes one of Nekrasov's instanton partition functions, namely the one with matter in the adjoint representation. Equivariant localization translates our formula into a rather surprising Pieri-type formula for Jack symmetric functions.

## 1.2

For brevity, we write  $\text{Hilb}_n = \text{Hilb}(S, n)$  when the surface  $S$  is understood. Let  $\mathcal{L}$  be a line bundle on  $S$ . Consider the virtual bundle  $\mathbf{E}$  over  $\text{Hilb}_k \times \text{Hilb}_l$  with fiber

$$\mathbf{E}|_{(I, J)} = \chi(\mathcal{L}) - \chi(I, J \otimes \mathcal{L})$$

over a pair  $(I, J)$  of ideal sheaves on  $S$  of colength  $k$  and  $l$ , respectively. Here

$$\chi(F, G) = \sum_{i=0}^2 (-1)^i \text{Ext}^i(F, G), \quad \chi(F) = \chi(\mathcal{O}, F)$$

for a pair of coherent sheaves  $F$  and  $G$  on  $S$ . Here and in what follows we assume that either  $S$  is compact or there exists a torus action on  $S$  such that the  $T$ -fixed loci are compact. In the latter case, the Ext-groups have finite dimensional  $T$ -eigenspaces, making the subtractions above meaningful.

By Riemann-Roch,

$$\text{rk } \mathbf{E} = k + l,$$

so we set

$$e(\mathbf{E}) = c_{k+l}(\mathbf{E}).$$

We define the operator  $\mathbf{W}(\mathcal{L}) = \mathbf{W}(\mathcal{L}, z) \in \text{End}(\mathcal{F})[[z, z^{-1}]]$  by its matrix elements

$$(\mathbf{W}(\mathcal{L}) \eta, \xi) = z^{l-k} \int_{\text{Hilb}_k \times \text{Hilb}_l} \eta \xi e(\mathbf{E}), \quad (2)$$

where the classes

$$\eta \in H^*(\text{Hilb}_k), \quad \xi \in H^*(\text{Hilb}_l)$$

are pulled back to the product via the natural projections and the inner product is the standard inner product on cohomology. In the noncompact

case, the integral is defined by equivariant localization. For each  $v \in \mathcal{F}$ ,  $W(\mathcal{L}) \cdot v$  is a formal power series in  $z, z^{-1}$  with coefficients in  $\mathcal{F}$  such that that coefficient of  $z^N$  is 0 for  $N \ll 0$ .

Note that one can always twist  $\mathcal{L}$  by a torus character, even when the torus action on  $S$  is trivial. The result is a scalar torus action on the fibers of  $\mathbf{E}$  and hence  $e(\mathbf{E})$ , as  $T$ -equivariant class, effectively includes all smaller Chern classes of  $\mathbf{E}$ . As we will see,

$$c_i(\mathbf{E}) = 0, \quad i > k + l. \quad (3)$$

Note also that Serre duality implies

$$W(\mathcal{L}, z)^* = (-1)^{\mathbf{N}} W(\mathcal{K} - \mathcal{L}, z^{-1}) (-1)^{\mathbf{N}}$$

where  $\mathcal{K}$  is the canonical bundle of  $S$ , the adjoint is with respect to natural inner products, and  $\mathbf{N}$  is the number-of-points operator, that is

$$\mathbf{N}|_{\mathcal{F}_n} = n \text{ Id}.$$

### 1.3

Our goal is to relate the operator  $W(\mathcal{L})$  to Nakajima operators on  $\mathcal{F}$  which are defined as follows. Assume  $l > k$ . Fix  $\gamma \in H_*(S)$  and consider the cycle

$$Z(\gamma) \subset \text{Hilb}_k \times \text{Hilb}_l$$

formed by pairs  $(I, J)$  such that  $J \subset I$  and the support of  $I/J$  is a single point of  $S$  lying on  $\gamma \subset S$ . By definition,

$$(\alpha_{k-l}(\gamma) \eta, \xi) = \int_{Z(\gamma)} \eta \xi, \quad \eta \in \mathcal{F}_k, \xi \in \mathcal{F}_l.$$

We will use the shorthand  $\alpha_{-n}(\mathcal{L})$  for the case when  $\gamma$  is the Poincaré dual of  $c_1(\mathcal{L})$ .

Now we can state our main result

**Theorem 1.**

$$W(\mathcal{L}) = \exp \left( - \sum_n \frac{(-z)^n}{n} \alpha_{-n}(\mathcal{L}) \right) \exp \left( \sum_n \frac{z^{-n}}{n} \alpha_{-n}^*(\mathcal{L} - \mathcal{K}) \right). \quad (4)$$

In particular, one can take  $k = 0$  in (2). The second factor in (4) fixes the vacuum vector

$$|\rangle = 1 \in H^*(\text{Hilb}_0) \subset \mathcal{F},$$

while  $\mathbf{E}$  becomes the tautological bundle  $\chi((\mathcal{O}/J) \otimes \mathcal{L})$ . In this case, (4) specializes to a formula of M. Lehn, see Corollary 6.6 in [4].

The product in (4) is a *vertex operator* in the sense that it may be uniquely characterized by its commutation relations with the Heisenberg algebra spanned by the Nakajima operators. It would be very interesting to find a geometric reason for this. Various connections and parallels between Hilbert schemes and vertex operator algebras may be found in the work of Li, Qin, and Wang, see e.g. [5].

Commutation relations for Nakajima operators yield the following

**Corollary 1.**

$$\text{str } q^{\mathbf{N}} W(\mathcal{L}) = \prod_n (1 - q^n)^{(\mathcal{L}, \mathcal{K} - \mathcal{L}) - e(S)},$$

where the supertrace is taken with respect to standard  $\mathbb{Z}/2$ -grading in cohomology and  $e(S)$  is the Euler characteristic of  $S$ .

## 1.4

In particular, if  $\mathcal{L} = \mathcal{O}$  then

$$\mathbf{E} \Big|_{\text{diagonal in } \text{Hilb}_n \times \text{Hilb}_n} = T \text{Hilb}_n,$$

in  $K$ -theory, see Proposition 2.2 in [1]. This explains why

$$\text{str } q^{\mathbf{N}} W(\mathcal{O}) = \prod_n (1 - q^n)^{-e(S)},$$

agrees with the generating functions for  $e(\text{Hilb}_n)$  obtained from Göttsche's formula [2].

Twisting  $\mathcal{O}$  by a torus character  $m$ , we get the Chern polynomial of the tangent bundle. Hence

$$\text{str } q^{\mathbf{N}} W(\mathcal{O}(m)) = Z_{\text{instanton}}(m),$$

where  $Z_{\text{instanton}}$  is the instanton part of Nekrasov partition function with adjoint matter of mass  $m$ , see [11]. In particular, Corollary 1 generalizes formula (6.12) in [11].

## 1.5

Recall that algebra  $\Lambda$  of symmetric functions over  $\mathbb{Q}$  is freely generated by the power-sum symmetric functions  $p_k$ , where  $k = 1, 2, \dots$ , that is,

$$\Lambda = \mathbb{Q}[p_1, p_2, p_3, \dots].$$

Consider the Jack inner product  $(\cdot, \cdot)_\theta$  on  $\Lambda$  with parameter  $\theta$ . This is the unique inner product such that

$$p_k^* = \frac{k}{\theta} \frac{\partial}{\partial p_k}$$

where  $p_k$  is viewed as the corresponding multiplication operator. We have  $\theta = 1/\alpha$ , where  $\alpha$  is the parameter used in Macdonald's book [7].

The integral form  $J_\mu$  of Jack symmetric functions is obtained by Gram-Schmidt orthogonalization of monomial symmetric functions with respect to  $(\cdot, \cdot)_\theta$ . It is normalized so that the coefficient of  $p_1^{|\mu|}$  in  $J_\mu$  equals 1.

Let  $E$  be the operator of multiplication by the sum of all elementary symmetric functions

$$E = 1 + e_1 + e_2 + \dots = \exp \left( \sum_n \frac{(-1)^n}{n} p_n \right).$$

and  $E^*$  the adjoint operator. Specialized to the equivariant cohomology of  $\text{Hilb}(\mathbb{C}^2, n)$ , formula (4) may be restated as the following Pieri-type formula for Jack polynomials:

**Corollary 2.**

$$(E^m (E^*)^{\theta-m-1} J_\lambda, J_\mu)_\theta = (-1)^{|\lambda|} \theta^{-|\lambda|-|\mu|} \times \prod_{\square \in \lambda} (m + a_\lambda(\square) + 1 + \theta l_\mu(\square)) \prod_{\square \in \mu} (m - a_\mu(\square) - \theta(l_\lambda(\square) + 1)). \quad (5)$$

Here  $a_\lambda(\square)$  and  $l_\lambda(\square)$  denote the arm- and leglengths of a square  $\square = (i, j)$  with respect to the diagram  $\lambda$ . These numbers are defined by

$$a_\lambda(\square) = \lambda_i - j, \quad l_\lambda(\square) = \lambda'_j - i,$$

where  $\lambda'$  is the transposed diagram.

We note that in the product (5) the ranges  $\square \in \lambda$  and  $\square \in \mu$  may be interchanged, see the proof of Lemma 3 below.

## 1.6

We expect to find a similar structure in the  $K$ -theory of Hilbert schemes. In particular, we expect a connection with the results of [8] where the operators defined by the virtual fundamental classes of the nested Hilbert scheme were described as certain  $(q, t)$ -analogs of vertex operators.

# 2 Proof

## 2.1

Let  $\Gamma(\mathcal{L})$  denote the right-hand side of (4) so

$$\mathbf{W}(\mathcal{L}) = \Gamma(\mathcal{L}) \tag{6}$$

is what we need to prove. As a first step, we reduce the general formula (6) to its special case, namely the case of the equivariant cohomology of a toric surface  $S$ .

Consider the universal ideal sheaf  $\mathcal{I}$  on  $\text{Hilb}_k \times S$  and the Künneth components of its Chern character

$$\sigma_i(\gamma) = \int_S \text{ch}_{i+2}(\mathcal{I}) \gamma \in H^{2i+\deg \gamma}(\text{Hilb}_k).$$

It is known that these classes generate  $H^*(\text{Hilb}_k)$  as a ring, that is, the monomials in these classes span  $H^*(\text{Hilb}_k)$ , see e.g. [5] and Section 6 in [4].

For given  $k$  and  $l$ , consider a matrix coefficient of  $\mathbf{W}(\mathcal{L})$  between two such monomials

$$\left( \mathbf{W}(\mathcal{L}) \prod \sigma_{p_i}(\eta_i), \prod \sigma_{q_j}(\xi_j) \right). \tag{7}$$

We may assume that the cohomology classes  $\eta_i$  and  $\xi_j$  have a well-defined parity.

Using the Grothendieck-Riemann-Roch formula, we may express (7) as an integral over

$$\text{Hilb}_k \times \text{Hilb}_l \times S \times S \times \cdots \times S \tag{8}$$

of a universal expression in characteristic classes of

$$\pi_{rs}^* \mathcal{I}, \quad r = 1, 2 \quad s \geq 3 \dots,$$

where  $\pi_{rs}$  is the projection on the  $r$ th and  $s$ th factor in (8), times a class  $\gamma_s$  of the form

$$\gamma_s \in \{\eta_i, \xi_j, \text{ch}(\mathcal{L}) \text{td}(S)\}$$

for each of the  $S$ -factors in (8).

The induction scheme of Ellingsrud, Göttsche, and Lehn, see Section 3 in [1], gives a universal evaluation of any such integral. The result has the form

$$\sum_{\{3,4,\dots\}=\sqcup G_i} \pm \prod_i \int_S C_{G_i} \prod_{s \in G_i} \gamma_s,$$

where the outer sum is over all partition of the  $S$ -factors in (8) into disjoint groups  $G_i$ , the sign accounts for a possibly different ordering of odd cohomology classes in the original and final integral, and each  $C_{G_i} \in H^*(S)$  is a characteristic class of  $S$ .

The characteristic classes  $C_{G_i}$  are universal, that is, independent of  $\gamma_i$ 's and therefore, the resulting formula for (7) is uniquely determined by the requirement that it holds in the equivariant cohomology of an arbitrary toric surface.

Similarly, using the calculus explained in Section 6 of [4], one derives analogous universal formulas for the matrix coefficients of  $\Gamma(\mathcal{L})$ . Thus the equality (6) in the toric case implies the equality in general.

For the same reason, it is enough to check the vanishing (3) in the toric setting.

## 2.2

Let a torus  $T$  with Lie Algebra  $t$  act on  $S$  with isolated fixed points. Equivariant localization provides the following isomorphism over  $F = \mathbb{C}(t^*)$ , the fraction field of  $R = \mathbb{C}[t^*]$ :

$$F \otimes_R \mathcal{F}_S = F \otimes_R \bigotimes_{s \in S^T} \mathcal{F}_{T_s S}.$$

Here each factor corresponds to the Hilbert scheme of points of  $\mathbb{C}^2 \cong T_s S$  with the inherited torus action.

Also from localization, we get the following factorization

$$W_S(\mathcal{L}) = \bigotimes_{s \in S^T} W_{\mathbb{C}^2}(\mathcal{L}|_s).$$

Furthermore,  $\Gamma(\mathcal{L})$  has a parallel factorization stemming from

$$\alpha_{-n}(\gamma) = \bigoplus_s \alpha_{-n}(\gamma|_s) ,$$

which reduces our theorem to the case  $S = \mathbb{C}^2$ .

For  $S = \mathbb{C}^2$ , the line bundle  $\mathcal{L}$  is a torus character  $\mathcal{O}(m)$ . The torus weight  $m$  has the meaning of mass in Nekrasov theory.

## 2.3

The  $F$ -linear space  $\mathcal{F}_{\mathbb{C}^2}$  may be identified with symmetric functions in such a way that Nakajima operators become the operators of multiplication by the power-sum symmetric functions, while the classes of torus-fixed points are mapped to properly normalized Jack symmetric functions. Since the character of the torus action in the fiber of  $\mathbf{E}$  over any fixed point is easily determined (see below), the statement of the theorem may be rephrased purely as a statement about symmetric functions.

Instead of attacking the symmetric function problem directly, we will first use geometric arguments to reduce it further to a simple special case.

## 2.4

Consider the following setup:

$$S = \mathbf{P}^1 \times \mathbf{P}^1, \quad T = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*, \quad \mathcal{L} = \mathcal{O}(m),$$

where  $(t_1, t_2, t_3) \cdot (z_1, z_2) = (t_1 z_1, t_2 z_2)$ , and  $m$  is a torus character of  $t_3$ .

Take two partitions  $\mu$  and  $\nu$  such that  $|\mu| = k$  and  $|\nu| = l$  and consider

$$w_{\mu, \nu} = w_S(\mu, \nu) = \left( \mathbf{W}(\mathcal{L}, 1) \prod \alpha_{-\mu_i}(L_1) | \rangle, \prod \alpha_{-\nu_i}(L_2) | \rangle \right) \in \mathbb{Z}, \quad (9)$$

where

$$L_1 = \{\text{pt}\} \times \mathbf{P}^1, \quad L_2 = \mathbf{P}^1 \times \{\text{pt}\}.$$

Let also  $w_{\mathbb{C}^2}^{[ab]}(\mu, \nu)$ ,  $a, b \in \{0, \infty\}$ , be the same expression over the Hilbert scheme of the chart of  $\mathbf{P}^1 \times \mathbf{P}^1$  by  $\mathbb{C}^2$  that contains the point  $(a, b)$ . Use the intersection of  $L_1, L_2$  with  $\mathbb{C}^2$  in place of  $L_1, L_2$ .



Since  $w_{\mu,\nu}$  is a integral of a top-dimensional class over a complete space, it is independent of the equivariant parameters (including  $m$ ) as well as the choice of the equivariant lifts of the classes  $L_1$  and  $L_2$ . Similarly, the number

$$w'_{\mu,\nu} = \left( \Gamma(\mathcal{L}, 1) \prod \alpha_{-\mu_i}(L_1) |\rangle, \prod \alpha_{-\nu_i}(L_2) |\rangle \right)$$

may be computed in either equivariant or ordinary cohomology and, in particular, is independent of equivariant parameters. We can use this independence as follows.

## 2.5

Let us compute  $w_{\mu,\nu}$  using equivariant localization. Each occurrence of the line  $L_1$  may be lifted to two different classes in equivariant cohomology, namely,

$$L_1^0 = \{0\} \times \mathbf{P}^1 \quad \text{or} \quad L_1^\infty = \{\infty\} \times \mathbf{P}^1,$$

and similarly for  $L_2$ . Such a lifting corresponds to a decomposition  $\mu = \mu^{[0]} \sqcup \mu^{[\infty]}$ ,  $\nu = \nu^{[0]} \sqcup \nu^{[\infty]}$ . Any choice of lifting allows us to compute  $w, w'$  by localization, so the answer must be independent of this lifting. In particular, using the same factorization of  $\mathbf{W}(\mathcal{L})$ ,

$$w_{\mu,\nu} = \sum_{\mu^{[00]}, \mu^{[0\infty]}, \mu^{[\infty 0]}, \mu^{[\infty\infty]}} \sum_{\nu^{[00]}, \nu^{[0\infty]}, \nu^{[\infty 0]}, \nu^{[\infty\infty]}} \prod_{a,b} w_{\mathbb{C}^2}^{[ab]}(\mu^{[ab]}, \nu^{[ab]})$$

where  $\mu_0, \mu_\infty, \nu_0, \nu_\infty$  correspond to any lifting,  $a, b \in \{0, \infty\}$ , and the sum is over terms such that  $\mu^{[a0]} \sqcup \mu^{[a\infty]} = \mu^{[a]}$ ,  $\nu^{[0b]} \sqcup \nu^{[\infty b]} = \nu^{[b]}$ . Using the parallel factorization for  $\Gamma$ , the expression above is also true when  $\mathbf{W}$  is replaced with  $\Gamma$ , and  $w$  is replaced with  $w'$ .

This formula for both  $w$  and  $w'$  gives an induction step for proving that  $w_{\mathbb{C}^2} = w'_{\mathbb{C}^2}$ , which is sufficient to prove the theorem. First, choosing the same equivariant lift (0 or  $\infty$ ) for each occurrence of  $L_1, L_2$  and solving for  $w_{\mathbb{C}^2}^{[ab]}(\mu, \nu)$  yields a function of  $w_S(\mu, \nu)$ ,  $w_{\mathbb{C}^2}^{[**]}(\mu', \emptyset)$ ,  $w_{\mathbb{C}^2}^{[**]}(\emptyset, \nu')$ , and lower order terms  $w_{\mathbb{C}^2}^{[**]}(\mu', \nu')$  with  $l(\mu') < l(\mu)$  or  $l(\nu') < l(\nu)$ . If  $l(\mu) > 1$  or  $l(\nu) > 1$ , then using a different equivariant lift gives  $w_S(\mu, \nu)$  as a function of the lower order terms  $w_{\mathbb{C}^2}^{[**]}(\mu', \nu')$ . We therefore have a formula for  $w_{\mathbb{C}^2}^{[ab]}(\mu, \nu)$  in lower order terms which also holds for  $w'_{\mathbb{C}^2}$ . The base cases  $\mu = \emptyset, \nu = \emptyset$  are covered by Lehn's theorem, so what is left is to prove the case  $l(\mu) = l(\nu) = 1$ .

## 2.6

In fact, we only need to check that

$$(\mathbf{W}(\mathcal{L})\eta, \xi) = (\Gamma(\mathcal{L})\eta, \xi)$$

for an arbitrary pair of classes

$$\xi \in H^*(\text{Hilb}(\mathbb{C}^2, k)), \quad \eta \in H^*(\text{Hilb}(\mathbb{C}^2, l)),$$

having nonzero inner product with the classes  $\alpha_{-k}|\rangle$  and  $\alpha_{-l}|\rangle$ , respectively.

To do this computation, we first need to set up the equivariant localization.

## 2.7

Let  $T \cong (\mathbb{C}^\times)^2 \subset GL(2)$  be the standard maximal torus, i.e.

$$\text{Lie } T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\}$$

In equivariant cohomology, we may identify  $\mathcal{F}_{\mathbb{C}^2}$  with symmetric polynomials over  $\mathbb{Q}[t_1, t_2]$  by requiring that  $|\rangle$  corresponds to 1 and

$$\alpha_{-n}(1) \mapsto p_n,$$

that is, the Nakajima operators become operators of multiplication by power-sum functions. It is known, see [13, 6] and also e.g. [12], that the classes of fixed points then correspond to properly normalized Jack polynomials. Concretely, take a monomial ideal

$$I_\lambda = (x_1^{\lambda_i} x_2^{i-1}) \subset \mathbb{C}[x_1, x_2].$$

Then

$$[I_\lambda] \mapsto t_2^{|\lambda|} J_\lambda \Big|_{p_i = t_1 p_i},$$

where  $J_\lambda$  is the integral Jack polynomial with parameter

$$\theta = -t_2/t_1.$$

## 2.8

Our next goal is to compute the character of the  $T$ -module

$$\mathbf{E}\Big|_{(I_\lambda, I_\mu)} = \chi(\mathcal{O}, \mathcal{O}) - \chi(I_\lambda, I_\mu).$$

Given a  $T$ -module  $V$  with finite-dimensional weight spaces, we denote by  $[V]$  its image in the representation ring of  $T$ . Concretely,  $[V]$  is represented by the trace of element  $(z_1, z_2) \in T$  in its action on  $V$ .

**Lemma 3.**

$$\left[ \mathbf{E}\Big|_{(I_\lambda, I_\mu)} \right] = \sum_{\square \in \mu} z_1^{-a_\mu(\square)} z_2^{l_\lambda(\square)+1} + \sum_{\square \in \lambda} z_1^{a_\lambda(\square)+1} z_2^{-l_\mu(\square)}. \quad (10)$$

*Proof.* We have

$$[\chi(I_\lambda, I_\mu)] = \frac{[I_\mu] [I_\lambda]^\vee}{[\mathcal{O}]^\vee}$$

where  $\checkmark$  denotes the dual module. Substituting

$$[I_\mu] = \sum_{i \geq 1} \frac{z_1^{-\mu_i} z_2^{1-i}}{1 - z_1^{-1}}, \quad [I_\lambda] = \sum_{j \geq 1} \frac{z_2^{-\lambda'_j} z_1^{1-j}}{1 - z_2^{-1}},$$

where  $\lambda'$  is the transposed diagram, into the above formula yields

$$\left[ \mathbf{E}\Big|_{(I_\lambda, I_\mu)} \right] = \sum_{i, j \geq 1} z_1^{j-\mu_i} z_2^{\lambda'_j-i+1} - \sum_{i, j \geq 1} z_1^j z_2^{1-i}.$$

Now observe the terms for which the exponent of  $z_1$  is  $\leq 0$  occur only in the first sum and correspond to the first sum in (10). The remaining terms may be determined using Serre duality which implies

$$\left[ \mathbf{E}\Big|_{(I_\lambda, I_\mu)} \right] = z_1 z_2 \left[ \mathbf{E}\Big|_{(I_\mu, I_\lambda)} \right]^\vee.$$

Note that the same argument with the roles of  $z_1$  and  $z_2$  interchanged yields the same formula except the ranges of summation  $\square \in \lambda$  and  $\square \in \mu$  get interchanged.  $\square$

The Lemma shows that (4) for the equivariant cohomology of  $\text{Hilb}(\mathbb{C}^2, n)$  is equivalent to the Pieri-type formula (5).

We also note that the character is a sum of exactly  $|\mu| + |\lambda|$  terms, implying the vanishing (3).

## 2.9

It is well known that Jack polynomials labeled by single row or column have nonzero inner product with the power-sum function of the same degree (this is true already in the Schur function case  $\theta = 1$ ). Therefore, as we saw in Section 2.6, it suffices to prove (5) in the special case

$$\lambda = (1^k), \quad \mu = (l).$$

Since multiplication by a function adds squares to diagrams and there are only two diagram fitting inside both  $\lambda$  and  $\mu$ , we have

$$\begin{aligned} (E^m(E^*)^{\theta-m-1}J_\lambda, J_\mu)_\theta = \\ (E^m, J_\mu)_\theta (E^{\theta-m-1}, J_\lambda)_\theta + \theta (E^m p_1, J_\mu)_\theta (E^{\theta-m-1} p_1, J_\lambda)_\theta, \end{aligned}$$

the factor of  $\theta$  in the second term coming from  $(p_1, p_1)_\theta^{-1} = \theta$ .

We have

$$(E^m, J_\mu)_\theta = \theta^{-l} \prod_{i=0}^{l-1} (m-i), \quad (E^m p_1, J_\mu)_\theta = l \theta^{-l} \prod_{i=0}^{l-2} (m-i),$$

which is an elementary statement about polynomials in one variable. Dually,

$$(E^m, J_\lambda)_\theta = \theta^{-k} \prod_{i=0}^{k-1} (m+i\theta), \quad (E^m p_1, J_\lambda)_\theta = l \theta^{-k} \prod_{i=0}^{k-2} (m+i\theta).$$

Replacing  $m$  by  $\theta - 1 - m$  in the above formula and summing up gives the right answer. This concludes the proof.

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